A New Electric Analogue Model for Nonsteady State Flow Problems

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A new, simple electric analogue model is demonstrated which gives solutions, accurate within ten %, to problems in nonsteady state flow of heat, diffusion, and flow of liquids in porous media. The analogue consists essentially of a sandwich of electrical conducting paper, polyethylene or polyester sheeting, and metal foil. One- or two-dimensional problems can be treated. This analogue provides a medium with distributed resistance and capacitance rather than the finite steps of conventional analogues; therefore two-dimensional problems of complex shape can easily be modeled. The analogue is pulsed by a square wave generator and the transient potential response is displayed on a cathode-ray oscilloscope.

Problems in nonsteady state heat conduction, diffusion, and fluid flow through porous media are often encountered in engineering design and operations. When these problems arise for simple geometrical shapes, the solutions usually can be found in the literature. For irregular or complex geometrical shapes, solutions can be obtained only by analogue models or high-speed digital computers. Pachkis (*1*) and Bruce (*2*) have described electric analogue models which can solve these problems, but their models are complex and expensive. Recently the trend has been toward the use of digital computers for the solution of nonsteady state flow problems. Digital-computer solutions usually require machine programs that are difficult and expensive to devise. Both the digital-computer solutions and most electric analogue models have a disadvantage because they require the continuous body under study to be transformed to a finite mesh. The process of making a "lumped" mesh from a continuum may introduce serious errors if there are regions of rapidly converging or diverging stream lines.

This paper describes a new, simple, continuous electric analogue model that is equivalent to an infinitely fine mesh and that will give an approximate solution of many one- and two-dimensional nonsteady state problems. The analogue consists of electrical conducting paper, a thin sheet of dielectric material, and metal foil all sandwiched together to form a continuous resistance and capacitance analogue. In its simplest form this analogue can model only homogeneous systems. In a slightly more complex form, it can model systems in which there are regions of different diffusivity constants*. A continuous variation of diffusivity constant with distance can also be modeled if the continuous variation in the prototype is transformed to a stepwise variation in the model. Variation of the diffusivity constant with time or potential cannot be modeled.

The paper analogue and the method of measuring potential transients described in this paper can model systems in which a potential is instantaneously applied or removed from any boundary or point or in which the applied potential is a known function of time.

The analogue can be constructed and the data taken and analyzed in less than a working day. The accuracy, about 10% is usually sufficient for engineering purposes.

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*The term *diffusivity constant* refers to the coefficient in the nonsteady state flow equation and is not limited to the coefficient in the diffusion equation.

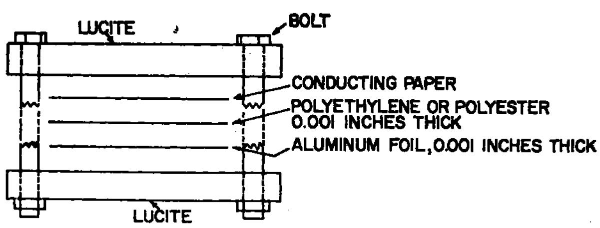

Fig. 1. Cross section of paper electric analogue model in exploded view. In actual model the bolts are drawn tight to force paper, polyethylene, and foil into contact.

PRINCIPLE OF ELECTRIC ANALOGUE MODELS

Problems in nonsteady state heat conduction, diffusion, and flow of liquids in porous media can be solved by an electric analogue model because all these phenomena are described by the same equation that governs flow of electricity in certain networks of resistors and capacitors.

The governing equation, usually called the heat equation, is in one dimension

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2} \tag{1}$$

and in two dimensions

$$\frac{\partial U}{\partial t} = \alpha\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) \tag{2}$$

DESCRIPTION OF PAPER ANALOGUE MODEL

The electric analogue model to be described here is in principle the same as the models of Pachkis (*1*) and Bruce (*2*), but it uses different kinds of resistance and capacitance elements and a different method of measuring transient voltage.

The electric analogue model described in this paper is made of Western Union L 39 Teledeltos or Timefax electrical conducting paper, 0.001 in. thick polyethylene or polyester sheeting (the kind used for packaging perishable food), and metal foil. The conducting paper, dielectric sheeting, and foil are sandwiched between two Lucite plates to form an analogue with a distributed resistance and capacitance medium. A typical analogue constructed during this study was made of two pieces of ½- by 6- by 6-in. Lucite. Ten ¼-in. holes were drilled along the edge to take the ¼ by 20 machine bolts that held the elements of

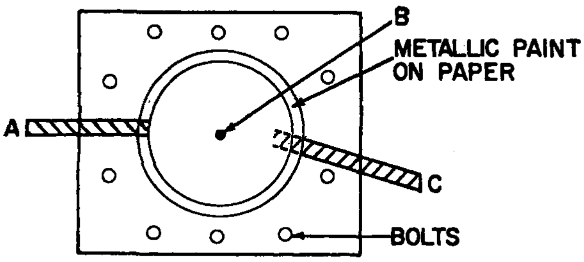

Fig. 2. Top view of paper analogue model of infinite cylinder. Point *A* is the lead to the outer edge, *B* is the lead to the center, and *C* is the lead to the ground plate.

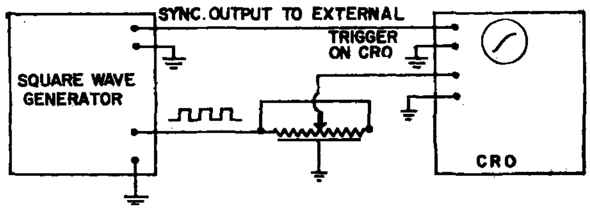

Fig. 3. Circuit diagram showing method of pulsing and measuring transient in paper analogue model.

the analogue in intimate contact. To be certain that there was uniform pressure on the entire surface of the conducting paper, dielectric sheeting, and foil combination, four *C* clamps were equally spaced around the center. A cross section of the analogue is shown in Figure 1. The top view of an analogue model representing a radial system is shown in Figure 2. Printed-circuit silver paint applied to the conducting paper acts as the electrode. Contact to the painted electrode is made by means of a strip of brass shim stock. The paint reduces to a negligible value the contact resistance between the paper and stock.

The resistance-capacitance product of the paper analogue model described above is so low that voltage build-up takes place in a few microseconds. The resistance of L 39 Teledeltos paper is 2,000 ohms/sq. in.; Timefax has a resistance of 6,000. The capacitance of the model is about 500 $\mu\mu$fd./sq. in., polyethylene or polyester 0.001 in. thick being used. If a model 1-in. wide and 3-in. long is pulsed at one end, the voltage at the other end will rise to 63% of the input voltage in about 4 μsec. when Teledeltos paper is used and in 12 μsec. when Timefax paper is used. The input pulse must be supplied by a high-speed switch or a square wave generator and the voltage build-up curve must be recorded photographically from a cathode-ray oscilloscope. An arrangement of this kind has been described by Lawson and McGuire (*3*).

The recently developed fast–rise-time oscilloscopes and the Polaroid oscilloscope camera make it convenient to obtain transient response directly. In the study reported here the analogue model was pulsed with a model 43A square wave generator manufactured by Electro-Mechanical Research, Incorporated, the transient response was displayed on a Tektronix model 545 oscilloscope, and the pattern on the oscilloscope was photographed with a Du Mont polaroid camera.

COMPARISON OF PAPER-ANALOGUE-MODEL SOLUTIONS TO ANALYTICAL SOLUTION

Paper analogue models were constructed to represent an infinite plate, an infinite square bar, and an infinite cylinder. A schematic sketch of the cylinder models is shown in Figure 2. Several models of each kind were made to determine the optimum size and combination of conductive paper and dielectric material. The infinite plate models were about 3 by 2 in., the square bar models were 2 to 4 in. on a side, and the cylinder models were 3 to 4 in. in diameter. The central contact was a 4–40 machine screw that was threaded through the upper Lucite plate and that made contact with the conducting paper through a small hole in the foil and dielectric. The potential at this contact, as a function of time, gave the central potential history of the model.

The circuit diagram showing the model, square wave generator, and oscilloscope is given in Figure 3. The square wave generator was operated at about 5,000 cycles/sec. This frequency is not critical. The only requirement is that there be sufficient time for the model to discharge between pulses. At 5,000 cycles/sec. the pulse length and discharge time are both 200 μsec. The model was essentially completely discharged after 20 μsec. The output impedance of the generator was 300 ohms (or less if an external resistor was shunted across the output), so that the model might discharge through the generator during the period between pulses.

The potential transient at the center of the model was picked up by a Tektronix P 450-L probe, which has an input capacitance of 2.5 $\mu\mu$fd. and an input resistance of 10 megohms. The high input impedance assured that the probe did not load the model. From the probe the signal was fed to the type 53/54K input unit of the Tektronix 545 oscilloscope. The fast rise time of this input unit, about 0.006 μsec., assured that the transient being measured was not distorted by the input amplifier characteristic.

The input to the model was displayed on the scope and photographed. The

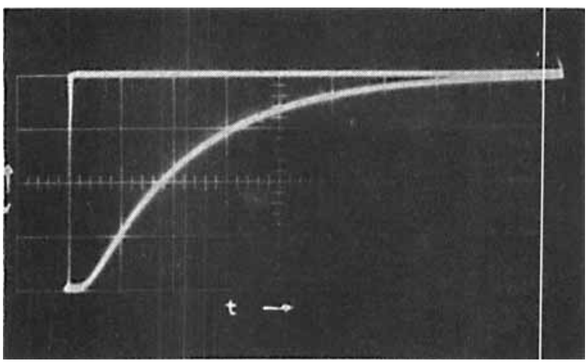

Fig. 4. Photoreproduction of trace displayed on oscilloscope. Both input impulse and transient are shown. Each large time division is 2 μsec. The maximum voltage is about 0.5 volt.

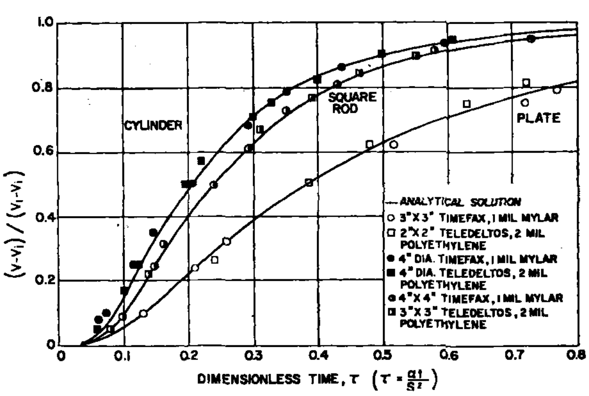

Fig. 5. Central potential history of several solids when the outer surface is suddenly raised from V_i to V_1. Lines are analytical solution; data points are from analogue model.

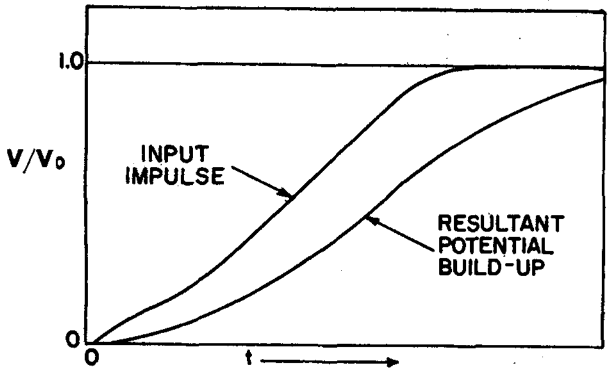

Fig. 6. Input impulse and resultant potential build-up curve.

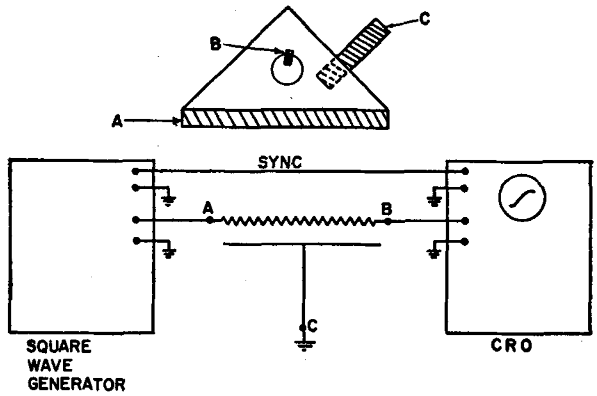

Fig. 7. Paper analogue model of insulated triangular bar and electrical circuit.

scope was then connected to the output and a second exposure was made on the same film. The result is shown in Figure 4. The grid is on a transparent plate that covers the scope. In Figure 4 each large division on the abscissa is 2.0 μsec.

The ordinate in Figure 4 is automatically normalized to unity by adjusting the input pulse height. The abscissa, however, is real time and must be converted to a dimensionless time in order to compare the model data with the analytical solution. The dimensionless time τ is given by $\tau = t/S^2RC$. In theory, R and C could be measured separately by standard measuring instruments. In practice, it was easier to calculate RC by comparing the model transient curve with the analytical solution for one model made of a given conducting paper and dielectric material. This value of RC was used to convert the real-time transient curve to a dimensionless time curve for all other models using the same conducting paper and dielectric. RC for Teledeltos paper and 0.002-in. polyethylene was 1.11×10^{-6} ohm farads/in.; for Teledeltos paper and 0.001-in. polyester (Mylar), RC was 2.20×10^{-6} ohm farads/in.; and for Timefax paper and 0.001-in. polyester, RC was 3.46×10^{-6} ohm farads/in.

Figure 5 shows the dimensionless central potential history as obtained by paper analogue models. RC was calculated for each conducting paper and dielectric combination by equating the model solution for an infinite plate at $(V - V_i)/(V_1 - V_i) = 0.50$ with the corresponding analytical solution as given by Jakob (*4*) or Schneider (*5*). The scatter of the model data points about the analytical curves is not much worse than that obtained by more complex analogue models (*5*, p. 353).

Timefax paper and 0.001-in.-thick Mylar polyester film gave models that had a potential build-up time at the center of about 10 μsec. This combination gave a transient curve that was least influenced by the oscilloscope input characteristics. On the other hand, it presented a highly capacitative load to the square wave generator with a resultant tendency to round off the corner of the input square wave. This rounding off was counteracted by shunting a resistor of from 20 to 100 ohms across the output of the square wave generator.

Teledeltos paper and 0.002-in.-thick polyethylene gave models that had potential build-up times of 1.5 to 2.5 μsec. The input characteristics of the scope, a 0.01-μsec. rise time, gave a slight distortion of the transient curve. However, this model did not significantly load the square wave generator. There seems to be little reason to believe any particular combination of paper and dielectric to be better than another.

SPECIAL APPLICATIONS OF PAPER ANALOGUE MODEL

Systems with Potential Gradient at Time Zero

The potential history of a system in which a potential is added to a potential gradient already existing at time zero is often desired. The potential history of such one-dimensional systems can be obtained from the paper analogue model if the original gradient is linear. In such cases it is necessary to add the nonsteady state solution obtained from an analogue, with no potential gradient at time zero, to the original gradient. A radial system can be treated in the same manner if the original gradient is linear with respect to the logarithm of the radius.

Systems with Zones of Different Diffusivity Constant

The paper analogue can be used to model systems in which there are regions of different diffusivity constant. For example, in a problem in fluid flow through a radial porous body in which there is a zone of reduced permeability around the central output hole, the diffusivity constant in the prototype is $k/\phi\beta\eta$, whereas in the analogue it is $1/RC$. In the zone of reduced permeability, therefore, $1/RC$ should be less than in the remainder of the analogue. The simplest way of reducing $1/RC$ in the analogue is to increase C. The analogue is therefore constructed with several thicknesses of polyethylene sheet as the dielectric over all the area except around the central output hole. In this area there is only one thickness of dielectric material. The ratio of permeability in the zone around the output hole to the permeability in the remainder of the system will be the ratio of the thickness of the dielectrics. If, for example, the prototype has a zone around the output hole one fifth as permeable as the remainder of the system, then one sheet of dielectric material is used for the area around the output and five sheets for the remainder.

Another situation that may arise is one in which the prototype has a diffusivity constant that is continuously variable with distance. The paper analogue can model such system if the continuous variation is converted to a stepwise variation. If, for example, a radial porous body has a continuously increasing permeability from the central hole to the outer edge, the paper analogue would be constructed so as to have the least thickness of dielectric material around the hole. The thickness of dielectric material would increase stepwise along a radius, thereby giving concentric rings which become thicker toward the outer boundary.

In any paper analogue models in which there is a stepwise change in thickness of the dielectric, it may be necessary to use a thin sheet of soft rubber under the metal foil in order to assure intimate contact of paper, dielectric, and foil at the step.

Calculation of Response to Any Arbitrary Input Function from the Response to an Instantaneous Pulse

If the potential build-up curve of any analogue is known for an instantaneous potential rise $A(t)$ at the input boundary, then the build-up curve $H(t)$ for the case when the potential rise is any arbitrary function of time $G(t)$ can be calculated by employing the superposition theorem given by

$$H(t) = \frac{d}{dt} \int_0^t G(t - \gamma)A(\gamma)\, d\gamma \qquad (3)$$

The integration can be carried out conveniently by numerical means for any specified value of time. Figure 6 shows an irregular input impulse and the resultant potential build-up for a model with transient response to a unit impulse typical of those responses shown in Figure 6.

EXAMPLES OF PROBLEMS THAT CAN BE SOLVED BY PAPER ANALOGUE MODELS

Several examples of engineering problems that can be solved by the paper analogue model are given below. The potential build-up curve is displayed on the oscilloscope and photographed as outlined previously.

Heat Flow

1. A long metal bar of triangular cross section with a circular hole running through the bar at the center is covered on two sides with insulating material. The surface of the center hole is also covered with insulation. The open side is placed at $t = 0$, on a hot plate of temperature T. The temperature at a point on the surface of the central hole as a function of time is desired. The initial temperature of the bar is uniform. The paper analogue and the electrical circuit are shown in Figure 7.

The RC product per unit area for the paper and dielectric material used for this analogue is calculated from a measurement on a suitable analogue whose solution is known, as described previously. The temperature build-up at B is calculated from RC and the potential transient at B.

Diffusion

A long hollow cylinder of a porous material is covered over part of its outer circumference by an impermeable layer. The cross section of this cylinder is shown in Figure 8. The cylinder is in a large vessel containing pure water. The porous material is saturated with pure water and the central hole is filled with pure water. At time $t = 0$ a salt solution is flowed through the central hole the salt concentration being kept constant in the hole. The problem is to find the salt concentration of the water in the outer vessel as a function of time. The paper analogue and the electrical circuit are shown in Figure 8. The condenser K represents the volume of the outer vessel and the capacity of K is calculated from the capacity of the analogue, the pore volume of the porous cylinder, and the volume of the outer vessel by the equation

$$\frac{\text{capacity of analogue}}{\text{capacity of } K} = \frac{\text{pore volume of porous cylinder}}{\text{volume of outer vessel}}$$

The capacity of the analogue is calculated from the RC product of an analogue of the same materials whose solution is known and from measured resistance from A to B.

The voltage build-up curve at point B is displayed on the oscilloscope. This gives V/V_∞ as a function of time at B. The salt concentration in the outer vessel at infinite time will be equal to the salt concentration of the solution flowing through the central hole. Therefore the salt concentration in the outer vessel at any time can be read directly, in fraction of the constant salt concentration in the central hole, from the potential build-up curve measured at B.

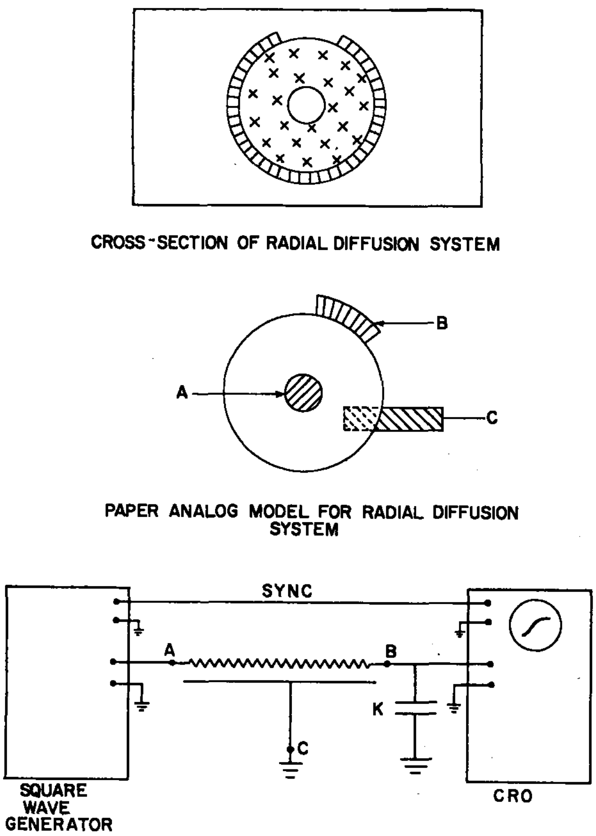

Fig. 8. Paper analogue model for radial diffusion.

SUMMARY

A simple electric analogue model has been devised for the solution of one- and two-dimensional nonsteady state flow problems. The model is composed of conducting paper, polyethylene or polyester sheeting, and metal foil. The model is pulsed by a square wave generator and the transient response is displayed on a cathode-ray oscilloscope.

Several problems are solved on the model and these solutions compared with the analytical solutions. The model gives results that do not deviate more than about 10% from the analytical solution. The method of applying the model to several nonsteady state flow problems is shown.

NOTATION

c = specific heat
C = specific capacity
D = diffusion coefficient
k = permeability
K = thermal conductivity
R = specific resistance
S = half thickness of plate or bar, or radius of cylinder
t = real time
T = temperature
U = temperature, concentration, pressure, or voltage
V = voltage
x = real distance
y = real distance

Greek Letters

α = diffusivity constant; $\alpha = K/C\rho$ for U = temperature, $\alpha = D$ for U = concentration, $\alpha = k/\phi\beta\eta$ for U = pressure, $\alpha = 1/RC$ for U = voltage
β = liquid compressibility
γ = an integrating variable
η = viscosity
ρ = density
τ = dimensionless time
ϕ = porosity

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Manuscript received on July 29, 1957; revision received on November 18, 1957; paper accepted November 20, 1957.